Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

# MODULE($S^2$): Algebra for Secondary Mathematics Teaching

Yvonne Lai & James Hart

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INSTRUCTOR VERSION



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## Part I

# How We Talk About and Explore Math

Welcome! We hope that this module will help give you opportunities to reflect on your own mathematical understandings – and how others may come to understand those ideas in different ways than you may have. In doing so, you will begin the process of thinking about mathematics so that you can understand your students' point of view and how to help them learn more deeply – whether in your future you are tutoring a niece or nephew or your own child, or you are a graduate student who is a TA, or you are a high school teacher, a middle school teacher, or an elementary school teacher.

**What this class looks like.** One common way that this module is used is as follows:

- At the beginning of each unit, you will work on an "Opening Inquiry". This inquiry will put into play some of the main concepts to be explored in the coming days.
- After reflecting on the ideas that surface from your work on the Opening Inquiry, you will work as a class to understand how these concepts are expressed formally and informally. Here, it can be good to reflect on why it is important to have **both** the formal and informal ways of thinking, not just one or the other, and why having both is important to students and teachers.
- In general, the module focuses on developing ideas through experiencing them in the form of problem solving.
- You are responsible for taking your own notes for the class. While some summary notes may be distributed each week, you should take notes on what is happening each lesson and organize them with any hand outs with in-class tasks so you can find things.
- Homework will be assigned on a weekly basis for most weeks. They will build on the ideas developed in class.

**Types of goals.** There are three overall kinds of learning goals in each lesson:

- Mathematical concepts – including definitions, theorems, procedures, and why they work.
- Mathematical/Teaching Practices – experiencing practices that cut across many aspects of mathematics or mathematics teaching, such as such as conjecturing, communicating with precision, leading a discussion, or introducing a mathematical procedure.
- Proof/Mathematical Structures – We use the phrase "Proof structures" to mean components needed in a proof of a certain kind of statement. E.g., showing membership in a set or showing that one set is a subset of another. The "components" are typically descriptions of the definition. E.g., "Showing $A \subseteq B$ requires showing that $x \in A$, then $x \in B$." We use the phrase "mathematical structures" to mean components needed to describe or define something. E.g., "To extend the definition of exponentiation from positive integer powers to powers of other real values means finding the definition for those other powers that is logically necessitated by the definition for positive integer powers and the properties of exponentiation."

**Overview of mathematical concepts**.

In Chapter 1, you will use the "parent relation" as a way to reflect on mathematical practices, learn how to work with each other, and gain intuition for sets, subsets, and negations.

In Chapter 2, you will see how the "parent relation" is an example of a relation. You will learn how to think about relations and functions in a way that connects middle school, high school, and university mathematical ideas. Here we will introduce functions and relations in general terms and then we will quickly specialize to the case where domain and codomain are subsets of real numbers.

In Chapters **??**, you will dig deeper into the concept of inverse relations. You will have an opportunity to practice teaching mathematical procedures to help students understand why they work.

**Key.**

**Instructor note.** Notes written in boxes are instructor notes. They are not visible in the student version.

A task given with a speech bubble indicates a task given orally. These are not contained in any worksheets, but they are visible in the student version of the text.

A task given inside grey brackets indicates a task that is provided in the *In-Class Resources* worksheets of each section. These worksheets are intended to be printed and distributed to students. There are also miniature versions of these worksheets in the student version.

[*Notes for solutions are often provided within the tasks. They are not visible in the student version.*]

**Structure of lessons.** Each lesson begins with an opening inquiry, then alternates between discussion of mathematics more formally and more tasks to explore ideas less formally. Each lesson ends with a summary. The reason for this summary is to help teachers process what just happened because sometimes it can be hard for learners to know immediately what it is that they have just experienced.

We have written some thoughts here that some piloters have found helpful:

- (A [Unit of Time] in the Life of Teaching the Algebra Modules) https://docs.google.com/document/d/1KC11eb0sb1pXRhJ4bPFn1qSWQjvj3x9m9tfi6lnZMBc/edit#heading=h.12egiwxwcyzm
- (Types of Learning Objectives in the Algebra Modules) https://docs.google.com/document/d/1Uc2OrjE2T1K89Y9N1ih0itM3dpZqXXW8LLdJEUiMMw0/edit#heading=h.tfkxj324wsda

**Things to keep in mind on the first day.** This first lesson is an important place to "set norms and expectations". This means communicating, both implicitly and explicitly, what productive conversation, exploration, questioning, and justification look and feel like. For instance, you may want to teach a class where:

- *Students embrace learning from their own individual and each others' work* – they view their own mistakes courageously and with an open mind; they accept that making errors and learning from them is a natural part of the mathematical process; they recognize what is worthwhile about others' reasoning and what needs further thought, and they do so constructively; they celebrate others' ideas.
- *Students view mathematical reasoning as the ultimate mathematical authority* – they have faith in their ability to learn to reason mathematically; they come back to the mathematics rather than to a perceived authority figure such as an instructor or a "smart" student to figure out what works; they seek precision in language while also understanding that going from informal language to precise language may take some time, may not happen right away, but is a valuable goal.
- *Students persist in seeking mathematical questions and answers* – they accept that setbacks are an important part of learning; they can work for an extended amount of time on one problem in productive ways; they celebrate when they do come to an understanding of a mathematical idea, especially one that is hard-won.

If these are values that you see a productive class expressing, there is much that you can do to foster these values beginning the first day. There are many different things you can do and say, and certainly different things may work better or worse for different instructors and different students. Here are some examples of things to do and say that have helped previous MODULE($S^2$)instructors:

- *Praise thoughtful errors.* It's easy to spot "right" answers and there can be a temptation to run with the way that some students have found exactly the "right" way to approach a problem. There is also a temptation to respond to "wrong" answers with saying matter-of-factly, "Not quite; what did others get?" But if you respond in these ways, and exclusively so as your form of interacting with students about their thinking, what message does that send to students about the role of mistakes in the process of working through mathematics? It may well send the message that the best work is the work that is correct the first try, or worse, that the most worthy students are those that only do correct mathematics and make no mistakes. Instead, an alternative approach is to look for thoughtful errors – the kind of thinking that is ultimately mathematically incorrect for some reason, but where thinking through the mistake has the potential to really get at something fundamental about the mathematics at hand or in the future. Moves that you can make to acknowledge thoughtful errors might include:

  - "I am so glad that you brought that up, [student name]. Did everyone understand what [student name] said? Can someone say in their own words what they understand of [student name]'s reasoning?" [If someone raises their hand to counter this idea] "Right now we're not interested in whether we agree or disagree with [student name], we are trying to understand what [student name] is thinking. What might they thinking? Why does it make sense to do this?"
  - "Let's see what happens when we follow this reasoning."
  - "We just learned a really important lesson about doing mathematics because of this reasoning. Thank you, [student name], for sharing your idea. This was incredibly helpful. Let's remember the lesson we learned throughout today and also as we move forward in this class."

- *Do not make a big deal when students get a correct answer right away. Focus on the process of getting to the answer, and on understanding the answer, rather than the answer itself.* The Fields Medalist William Thurston (1994) observed of his colleagues, "I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than the knowledge, people want *personal understanding*." (p. 51, emphasis by Thurston). The same is true of students, or at least we would like to be a truth about students. Moves that emphasize understanding over the answer might include:

  - (As a matter-of-fact first reaction to the correct answer) "You answered $X$. What was your reasoning for that answer?" . . . "What do others think of this reasoning?"
  - "[Student name] arrived at the solution $X$, and just shared their reasoning. Did anyone else arrive at this solution? Did you have similar reasoning or different reasoning?"
  - "Let's think back on why this answer makes sense."

- *Relinquish your authority to the students and the mathematics.* A common question instructors hear is, "What do you want?" or "Is this what you are looking for?" Sometimes the answer to these questions really does rest with you, the instructor – especially if it is about specific directions that you are setting for your students that can't be derived from mathematical reasoning. However, answering these questions from your authority as an instructor can be less useful if the questions are actually about mathematical reasoning, for instance, if the question is about whether a proof or solution is correct. In these cases, it can be more productive to return the responsibility of these questions to the students and the mathematics:

  - "Can you tell me more about how you arrived at this?"
  - "Tell me about what's here."
  - "How does this help to give a solution to the question we are working on?"
  - "How complete do you think it is?" . . . "What about your work are you sure about, and what are you less sure about?"

- *Give students ways to work constructively with each other.* Working with each other on mathematics is not necessarily a natural skill; it is a learned skill. Help your students find ways to talk to each other about their thinking. While students are working, stir the pot (meaning, find ways to provoke productive disagreement and/or discussion).

  - "I see that [student A] and [student B] have different answers. It looks like you have something to resolve. [Student A] and [Student B], will you share how you did your work with each other and figure out what's really going on?"
  - "I see that [student A] and [student B] have arrived at the same answer, but it looks like you've done it in different ways. Will you compare what you've done and see how they match each other or do not?"
  - "It looks like [student A] has drawn a graph and [student B] has used mostly equations. Are you thinking about the same thing? Will you talk to each other about how your thinking matches up or not?"
  - "It looks like [Student A] worked on [Case 1] whereas [student B] worked on [Case 2]. Are there more cases to consider? Are both cases necessary? You should talk to each other to figure this out."

# 1 Sets and Conjectures (Week 1) (Length: 2.5 hours)

## Overview (For Instructors)

In this section, we will quickly describe the key ideas presented in this activity. Instructors may choose to share these notes with students, especially after the activity is completed.

**Content**

**"Parent" relation**, implicitly defined as a relation which assigns elements of $\mathbb{N}$ to its factors; used to examine subsets, mathematical statements and their negations, properties of $\mathbb{R}$ and $\mathbb{Z}$, and to engage in mathematical practices.

(Looking ahead:) The parent relation is used in Section 2 to introduce relations and inverse relations.

**Subset**, **superset**, **strict subset**, and **strict superset**; **equality of sets** $A$ and $B$, defined as $A \subseteq B$ and $B \subseteq A$.

**Mathematical statements**, defined as those which can be evaluated as true or false; and

**Negation** of mathematical statement $S$, defined as a statement which is false if and only if $S$ is true.

**Properties of $\mathbb{R}$ and $\mathbb{Z}$** assumed. (These may have been introduced previously in an abstract algebra course.)

**Proof Structures**

**To show that $x \in A$** means showing that $x$ satisfies set membership rules for $A$; and **to show that $x \notin A$** means showing that $x$ does not satisfy at least one set membership rule of $A$.

**To show that $A \subseteq B$** requires showing that if $x \in A$, then $x \in B$.

**To show that $A \subsetneq B$** requires showing that: (1) $A \subseteq B$; (2) there is an element of $B$ that is not in $A$.

**To show that $A = B$** requires showing that: (1) $A \subseteq B$; (2) $B \subseteq A$.

**Mathematical/Teaching Practices**

**Clarifying mathematical questions**, meaning to determine how different interpretations of question statements may have different mathematical consequences.

**Conjecturing and being precise**, in the sense of giving "satisfying" answers to mathematical questions

**Communicating proofs well**, which includes specifying claims, the body of the proof, and givens and conclusions explicitly, clearly, and correctly.

## Summary

We introduce the "parent relation" as a context for engaging in mathematical practices as well as learning how to work with each other on exploratory tasks. The main tasks in this lesson are:

- *Which numbers have more than one pair of parents?*
- *Is one of these sets a subset of the other set? Check the mathematically correct statements. If you put a check in the $A \neq B$ column, list an element that is in one but not the other.*

| | $A \subseteq B$ | $A \subsetneq B$ | $A \supseteq B$ | $A \supsetneq B$ | $A = B$ | $A \neq B$ | Neither is subset of the other |
|---|---|---|---|---|---|---|---|
| $A =$ multiples of 3, $B =$ multiples of 6 | | | | | | | |
| $A =$ multiples of 6, $B =$ multiples of 9 | | | | | | | |
| $A = \{n^2 : n \in \mathbb{N}, n \geq 1\}$, $B = \{1 + 3 + \cdots + (2n-1) : n \geq 1\}$ | | | | | | | |
| $A = \{$Functions equivalent to $f(x) = a(x+h)^2 + v$, where $a, b, v \in \mathbb{R}\}$, $B = \{$Functions equivalent to $g(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}\}$ | | | | | | | |

Along the way we introduce notation for sets and subsets, discuss mathematical statements and their negations, and describe properties of $\mathbb{R}$ and $\mathbb{Z}$ assumed for now. There are also tasks in this lesson addressing these ideas.

*Acknowledgements.* The structure and some tasks are from notes from Mira Bernstein and used with permission.

**Materials.**

- Handouts in In-Class Resources (can be printed double-sided)
- Colored chalk / markers to highlight different parts of good proof communication

Now, let's get started!

**Instructor note.** Distribute handout with this question. As teachers work on it, circulate and listen to the questions and comments they make as they work through the questions. They may say and do things that will lead into a discussion on clarifying the question, precision, and also what it means to have less or more satisfying answers to a question.

# Opening inquiry: Number parents

**Instructor note.** Hand out Activity 1 and have students work together in small groups for a few minutes.

**Activity 1 — Number Parents**

Two numbers $a$ and $b$ are *parents* of another $c$ if $c = ab$. We will call $c$ the *child* of $a$ and $b$. Let's assume that a child cannot be one of its own parents.

Q1. What pairs of numbers are excluded from being parents?

Q2. A student tells you that the number 12 has exactly two sets of parents, namely $\{2, 6\}$ and $\{3, 4\}$. Do you think this student is correct?

Q3. Another student says that $\{5, 12/5\}$ is also a set of parents for 12. How would you respond to this student?

Q4. Are there any numbers that have an infinite number of parent sets?

Q5. If you only want numbers to have a *finite* number of parent sets, what changes would you make in the definition?

**Instructor note.** The point of these initial questions is to help students realize that they can read more into a definition than is actually present. Most students will assume that "number" means "positive integer". If they are not cognizant that they are using this assumption, they may not recognize that the answer to Question 2 is "NO" while the answers to Questions 3 and 4 is "YES". Most students will conclude that the definition should be modified by replacing "number" with "integer" (or even "positive integer") but be ready to explore other suggestions.

**Instructor note.** The parent relation can be formally defined as the relation that assigns a natural number to its nontrivial factors. This definition will be given later in the module, though it is not given yet. This might seem like a strange decision for a math course: after all, definitions play a pretty important role! But it is because definitions play such a huge role – and that students of all ages, including undergraduates, don't always understand the role of formal definition – that we wait on the formal definition.

We want teachers to appreciate the work that it can take to go from a mathematical query, especially one that uses ideas they are familiar with and where there is some ambiguity, to figure out what needs to be defined and why. The point is for for the teachers to work out the finer points or ambiguity, since that is something that they will want to work with their students on and it is something that it would be good for them to do for themselves.

**Instructor note.** Hand out Activity 2 and have students work in small groups. Give groups a chance to rank the answers and answer Question 7, then have them display their rankings without justification. Let the various groups compare rankings, then engage the full class in a discussion of how "satisfying" should be defined.

**Activity 2: Refining Number Parents**

OK, now let's refine our Number Parent relation by restricting attention only to positive integers. Working under this refinement, let's consider the following question:

- Which positive integers have more than one parent set?

Q1. Here are some possible answers (without explanations) to this question that prospective teachers in previous courses have given. On a 1 to 5 scale, rank these answers as least *satisfying* (1) to most *satisfying* (5).

   A1. Any number with at least least three different factors has more than one pair of parents.

   A2. 12 has more than one pair of parents.

   A3. Any number with at least three different factors (other than itself or 1) has more than one pair of parents. There are no other numbers with more than one pair of parents.

   A4. 12, 18, 20, 28, 30, 42, 44 each have more than one pair of parents.

   A5. Any number with at least three different factors (other than itself or 1) has more than one pair of parents.

Q2. What criteria did you use in deciding how satisfying each answer is?

**Instructor note.** If the discussions do not seem fruitful, suggest the following question:

- If a child asks you a question, what would make your answer "satisfying" to that child?

The key point that should be pulled from the discussion is that an answer is satisfying when it gives the most complete and correct understanding of a situation *for the given audience* .

Once the criteria for "satisfying" are settled, have the groups revisit their rankings (possibly changing them) and prepare to justify. Here is a sample ranking with justifications.

- Answer [A1] may seem fairly satisfying, but it is slightly incorrect. Consequently, this answer would be the least satisfying.
- Answers [A2] and [A4] should rank near the bottom because they don't give any sort of pattern or big picture of what's going on. They also raise the question: "Are those the only ones?"
- Answer [A5] is getting close, but still, raises the question of whether there are more answers.
- Answer [A3] is the most satisfying because it provides the big picture of when a number works, and also says, yes, these are the only answers.

# Reasoning with Sets

**Instructor note.** Hand out Activity 3 and have students work in small groups. Don't spend too much time on Questions 1 and 2. The best approach would be to have groups discuss Question 1 for two or three minutes, then take a show of hands as a vote on the truth-value of each statement. for each part. Note those parts where there is a difference of opinion. Select a student voicing each opinion and have that student justify their decision. Repeat this process for Question 2, not taking more than about fifteen minutes for both questions together.

**Activity 3, Part 1 — Reasoning with Sets**

When we are trying to answer a mathematical question, or when we are trying to improve an answer to a mathematical question, we are engaging in the process of *conjecturing*. A *conjecture* is a statement that arises from this process. A conjecture becomes an *answer* to the mathematical question once we are able to give a satisfying justification for it.

**Definition 1.1.** A *set* is any collection of objects. We typically use a capital letter like $S$ to represent a set. The objects constituting the set are called *members* or *elements* of the set. If $a$ is an element of a set $S$, it is common to write $a \in S$; and it is common to write $a \notin S$ when $a$ is *not* an element of $S$.

Q1. When a set $S$ consists of just a few members, we often define the set simply by listing the members, separated by commas and enclosed in brackets. For example, consider the set $S = \{1, 2, \{3, 4\}, \{5\}\}$ along with the mathematical question "What objects are elements of the set $S$?" Which of the following conjectures do you think are true?

Circle the true statements, and cross out the false statements. Then give justifications for your decisions.

(a) We have $2 \in S$.
(b) We have $\{3, 4\} \in S$.
(c) We have $3 \in S$.
(d) We have $\{2\} \in S$.

[*Solution. Statements (a) and (b) are TRUE, while Statements (c) and (d) are FALSE. There may be a difference of opinion on Statement (b), and there will likely be a difference of opinion on Statement (c) and (d).*]

Q2. We say that a set $A$ is a *subset* of a set $B$ provided every element of $A$ is also an element of $B$. We write $A \subseteq B$ in this case. Consider the set $S$ from Question 1 along with the mathematical question "What sets are subsets of $S$?" Which of the following conjectures do you think are true?

Circle the true statements, and cross out the false statements. Then give justifications for your decisions.

(a) We have $S \subseteq S$.
(b) We have $1 \subseteq S$.
(c) We have $\{1\} \subseteq S$.
(d) We have $\{5\} \subseteq S$.
(e) We have $\{\} \subseteq S$.

[*Solution. Statements (a), (c), and (d) are TRUE, while Statements (b) and (c) are FALSE.*]

**Note:** One helpful metaphor may be thinking of the braces (the { and } ) as permanent packaging, like gift wrap that doesn't come off. You can't take out what's inside the packaging. You can only hold the whole package. Even if only one thing is wrapped, you still can't hold the thing by itself, you can only hold it with its gift wrap. But if an object is not wrapped, you can hold that object by itself.

The set that contains no elements is called the *empty set*. The empty set can be denoted by $\{\}$; it is also customary to let $\emptyset$ represent this set.

**Instructor note.** Activity 3 Part 2 provides an example of reasoning with the empty set; its rationale is to make the point that, when reasoning with the empty set, we can have seemingly false statements that we take to be logically true.

**Activity 3, Part 2 — Reasoning with Sets, continued**

Q1. Think about the set $A$ consisting of all members of this class, along with the mathematical question "Who is an element of the set $A$?" Is the following conjecture TRUE or FALSE?

- All students in this class who are under 5 years old are also over 100 years old.

Justify your answer.
[*Solution. The statement is TRUE. In particular, this statement is TRUE because there are no members of $A$ that are less than five years old. Let students debate the truth value of this statement for a short time. It may help to have students think about the statement as a "promise" — If you choose any member of this class who is less than five years old, then I promise you this member is also over 100 years old. The "promise" cannot be broken, because you cannot choose such a member.*]

When a set $S$ contains a lot of elements, we often define the set by providing a generic letter name for its members along with a collection of rules that allow us to determine whether a particular object is an element of $S$. The object name and the defining rules are separated by a colon or a vertical line, and the whole thing is enclosed in brackets. For example,

$$B = \{x \mid x \in A, \text{ and } x \text{ is less than 5 years old, and } x \text{ is more than 100 years old}\}$$

defines the set of all members of our class who are both less than five years old and greater than 100 years old. This formatting is called *set-builder notation*.

**Activity 3, Part 3 — Reasoning with Sets, continued**

Q1. We say that a set is *well-defined* when it is possible to determine exactly which objects are members of the set. Are any of the following sets well-defined?

(a) $X = \{y \mid y \text{ is an even, positive integer}\}$

(b) $Y = \{a \mid a \text{ is a person living close to Baltimore}\}$

(c) $Z = \{x \mid x \text{ is not a person}\}$

[*Solution. Sets X and Z are both well-defined, while set Y is not. (What do we mean by "close"?) Let students discuss these sets in small groups for a few minutes and then take a show of hands for each set. Note any differences of opinion. Let students justify their answers.*]

Q2. We say that a set is *well-founded* provided the set does not have itself as a member. Is the set $Z$ from Question 4 well-founded? Explain your thinking.
[*Solution. Since the set Z is not a person, we must conclude that $Z \in Z$. Therefore, this set is not well-founded. Again, let students ponder this question for a few minutes in small groups, then ask for a show of hands. Students may not think about applying the non-person test to an abstract concept like a set, so it may be necessary to prompt them with a question like"Is Z a person?"*]

**Instructor note.** Well-foundedness is a weird concept, but it is critical in mathematical disciplines such as universal algebra and category theory. Indeed, most mathematicians do not consider non well-founded collections to be sets at all, preferring to call such things "proper classes" and treat them with great logical care. Outright contradictions can arise from treating non well-founded collections like "the collection of all sets" as sets. For example, it is a routine exercise in function theory to prove that the cardinality of *any* set is strictly smaller than the cardinality of its powerset (collection of all subsets of the set). However, the "set" $S$ of all sets must contain $S$ as a member; and this forces the cardinality of the powerset of $S$ to be equal to the cardinality of $S$.

# Proving Conjectures About Sets

## Subset exploration

**Instructor note.** Hand out Activity 4 and let students work in small groups for a few minutes on Question 1. Rather than merely asking for correct answers, ask students to justify their decision based on the rule of set membership. After going over Question 1, assign Question 2 individually. Direct students to work individually on Row 1 and Row 2 for a few minutes. Then go over those before assigning Rows 3-4.

**Activity 4 — Describing Sets**

**Proof Structure: Showing set membership.** To show that $x \in S$ means showing that $x$ satisfies set membership rules for $S$; to show that $x \notin S$ means showing that $x$ does not satisfy at least one set membership rule of $S$.

Q1. It is common to let $\mathbb{Q}$ represent the set of rational numbers. Let

$$S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}.$$

Are the following statements TRUE or FALSE? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $-1 \in S$

[*Solution. TRUE / FALSE / TRUE / TRUE.* 0.5 *can be written as the fraction* $\frac{1}{2}$, *and* $|0.5| < 2$. *The number* 0.5 *satisfies all the rules of membership of S, so it is an element of S. Although* $3.5 = \frac{7}{2}$, *it does not satisfy the condition* $|x| < 2$. $|0.25| < 2$ *and* $0.25 = \frac{0.5}{2}$. $|-1| < 2$ *and* $-1 = \frac{-2}{2}$.]

Q2. Is $A$ a subset of $B$ or vice versa? Complete this table with "yes" or "no" in each cell.

**Instructor note.** Go over Rows 1-2. Different interpretations lead to different conclusions. Discuss why. From now on assume: The word "multiple" refers to "integer multiple". Then assign Rows 3-4. An optional but interesting follow-up question would be to ask "Do we have set equality if we restrict the constants to be integers instead of real numbers?"

| | $A \subseteq B$ | $A \subsetneq B$ | $A \supseteq B$ | $A \supsetneq B$ | $A = B$ | $A \neq B$ | Neither is subset of the other |
|---|---|---|---|---|---|---|---|
| $A = \{n \in \mathbb{Z} : n \text{ is a multiple of } 3\}$, $B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 6\}$ | | | | | | | |
| $A = \{n \in \mathbb{Z} : n \text{ is a multiple of } 6\}$, $B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 9\}$ | | | | | | | |
| $A = \{n^2 : n \in \mathbb{N}, n \geq 1\}$, $B = \{1 + 3 + \cdots + (2n-1) : n \geq 1\}$ | | | | | | | |
| $A = \{$Functions equivalent to $f(x) = a(x+h)^2 + v$, where $a, b, v \in \mathbb{R}\}$, $B = \{$Functions equivalent to $g(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}\}$ | | | | | | | |

We found that there were several ways that these questions needed to be clarified:

In Row 1, we asked: what kind of multiples? We decided to consider only integer multiples. In Row 3, we asked: How do we decide whether two functions are equal? It also helped to have students construct example functions from each set and then figure out how to transform the representative from $A$ into a representative from $B$ and vice-versa.

**Instructor note.** Discuss the ideas of writing conjectures as if/then statements (leading up to Q1 and Q2). Then distribute handout and ask students to work on Q1 and Q2.

**Activity 5 — A Sample Proof, Part 1 (Intro)**

In logic, a *statement* is a declarative sentence that is either TRUE of FALSE but not both (or neither).

**Definition 1.2.** A *conjecture* about a mathematical process is a statement whose truth-value can be tested using the rules we have assumed in the process. Conjectures are conditional (IF-THEN) statements, but they may not always be phrased that way.

We don't always write conjectures explicitly as conditional statements, but with enough care, it is always possible to rephrase a conjecture in the form

IF [Statement 1] THEN [Statement 2].

Written in this form, Statement 1 is called the *hypothesis* and Statement 2 is called the *conclusion* of the conjecture.

We can think of a conjecture as a "promise." We *promise* that *IF* Statement 1 is true, *THEN* Statement 2 must also be true. There is only one situation in which a promise is broken — when Statement 1 is true *and* Statement 2 is false.

When we *prove* a conjecture, we are presenting a logically sound argument which demonstrates that *it is impossible to break the promise*.

Q1. The following statements are conjectures about the set $\mathbb{Z}$ of integers. How could you write these conjectures as IF-THEN statements?

[Statement 1.] Every integer that is a multiple of 4 is also a multiple of 2.

[Statement 2.] There exists an integer that is both even and odd. (It helps to consider the set $B$ of all integers that are both even and odd.)

[*Solution. Statement 1 can be rephrased as "IF an integer is a multiple of 4, THEN it is also a multiple of 2." Statement 2 is trickier. One way to rephrase would be "IF E is the set of all even integers and O is the set of all odd integers, then* $E \cap O$ *is nonempty."*]

**Instructor note.** Discuss universal and existential conjectures. Explain why Statement 1 is universal, then discuss the proof structure for subset proofs. Pose Q2 on the board. Go over Q2. Say that Part 2 of this Activity 5 summarizes the discussion just taken, and in Part 3 we will apply the ideas to our conjecture. Then distribute Activity 5 Parts 2 and 3.

**Activity 5 — A Sample Proof, Part 2 (Proof Structure for Showing Subsets)**

Statement 1 above is an example of a *universal* conjecture. Universal conjectures claim one set is a subset of another. (Statement 1 is a claim is that the set of integer multiples of 4 is a subset of the set of integer multiples of 2.)

Statement 2 above is an example of an *existential* conjecture. Existential conjectures claim that a particular set is nonempty. (Statement 2 is a claim is that the set of all integers that are both even and odd is nonempty.)

Let's consider Statement 1 — Every integer that is a multiple of 4 is also a multiple of 2.

If we want to prove this conjecture, then we have to construct a logically sound argument that shows letting "$X$ be the set of all integer multiples of 4 and $Y$ be the set of all integer multiples of 2" *guarantees* that $X \subseteq Y$.

**Proof Structure for Showing Proper Subsets**

**Showing** $X \subseteq Y$.

- To show that $X \subseteq Y$, you begin with an arbitrary $a \in X$. You then use the membership rules for $X$ and $Y$ to explain why $a$ is also a member of $Y$.

*Proof.* Suppose $a$ is a member of the set $X$. This tells us that $a$ is an integer multiple of 4. Therefore, we know that $a = 4n$ for some integer $n$. Now, we know $4 = 2 \cdot 2$, so the fact that integer multiplication is associative lets us conclude

$$a = 4n = (2 \cdot 2) \cdot n = 2 \cdot (2 \cdot n)$$

Of course, $2n$ is also an integer; hence we may conclude that $a$ is an integer multiple of 2. This tells us that $a$ satisfies the criteria for membership in the set $Y$. Consequently, we know that $a \in Y$.

Since we have shown that an arbitrary member of $X$ is also a member of $Y$, we may conclude that $X \subseteq Y$, as desired. □

**Definition 1.3.** We say that a set $Y$ is a *proper* subset of a set $X$ provided every element of $Y$ is a member of $X$, but there are members of $X$ which are *not* members of $Y$. It is common to write $Y \subsetneq X$ in this case.

Here is a conjecture we could make using this notion:

- If $A$ is the set of all integer multiples of 3 and $B$ is the set of all integer multiples of 6, then $B \subsetneq A$.

Q2. Is this conjecture universal or existential? Justify your thinking.

[*Solution. This conjecture actually makes two claims — one that is universal and one that is existential. First, the conjecture claims that every integer multiple of* 6 *is also an integer multiple of* 3*. Second, the conjecture claims that there exist integer multiples of* 3 *that are not integer multiples of* 6.]

**Activity 5 — A Sample Proof, Part 3 (Applying the Proof Structure)**

Q3. Let's think about how we could go about proving this conjecture. Let's let $A$ be the set of all integer multiples of 3 and let $B$ be the set of all integer multiples of 6. We really have two claims to verify. What does each part mean?

**Proof Structure for Showing that Something is a Proper Subset**

**Part 1: Proving that every member of $B$ is a member of $A$.**

Notes on what this means:

[*Solution:*

- *To show that $B \subseteq A$, you begin with an arbitrary $a \in B$. You then use the membership rules for $A$ and $B$ to explain why $a$ is also a member of $A$.*
- *Here, this means showing that every multiple of 6 is also a multiple of 3.*

]

**Part 2: Proving there exist members of $A$ that are not members of $B$.**

Notes on what this means:

[*Solution:*

- *You use the membership rules for $A$ and $B$ to identify a specific element of $A$ that is not an element of $B$.*
- *So you need to find a multiple of 3 that is NOT a multiple of 6.*

]

**Instructor note.** Let the class discuss this question. Move about the class and listen to the group conversations, asking the groups to identify some key aspects of the student's proof.

**Activity 5 — A Sample Proof, Part 4 (Looking at a sample proof)**

Let's look at a student's proof from a previous offering of this course.

Claim. $A = \{3n : n \in \mathbb{Z}\}$, $B = \{6n : n \in \mathbb{Z}\}$ $\Rightarrow B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$, $B = \{6n : n \in \mathbb{Z}\}$.

① $B \subseteq A$ We show: $x \in B \Rightarrow x \in A$.

Given $x \in B$.

(*)
$x = 6k$, $k \in \mathbb{Z}$ — by defn of membership in $B$
$= 3 \cdot 2k$
$= 3n$, $n \in \mathbb{Z}$. — by closure of mult in $\mathbb{Z}$ ($2, k \in \mathbb{Z} \Rightarrow 2k \in \mathbb{Z}$)

Hence $x$ satisfies membership rules for $A$
$\Rightarrow x \in A$.
By defn of subset, $B \subseteq A$. ☐1

② $\exists x \in A$ s.t. $x \notin B$ We find an elt of $A$ not in $B$. Observe that all $x \in B$ are even (by defn of even):

$x \in B \Rightarrow x = 3 \cdot 2k = 2 \cdot 3k$ — by ⊛ and comm of mult in $\mathbb{Z}$
$= 2m$, $m \in \mathbb{Z}$ — by closure of mult in $\mathbb{Z}$ ($3, k \in \mathbb{Z} \Rightarrow 3k \in \mathbb{Z}$)

But there are members of $A$ that are odd, e.g., 3, 6, 15, . . . These members of $A$ are not in $B$. ☐2

$B$ and $A$ satisfy the defn of strict subset $\Rightarrow B \subsetneq A$. ☐

Q4. Look over this proof carefully.

- Has the student addressed the two steps necessary to verify that $B$ is a proper subset of $A$?
- What is the students' reasoning for each part?
- What is done well? What could be clearer?

**Instructor note.** Distribute Part 5. Let the class discuss this question. The diagram below highlights some key features. After discussion, hand out Part 6.

**Activity 5 — A Sample Proof, Part 5 (Features of Proof Communication)**

While you are looking over any proof, think about the following.

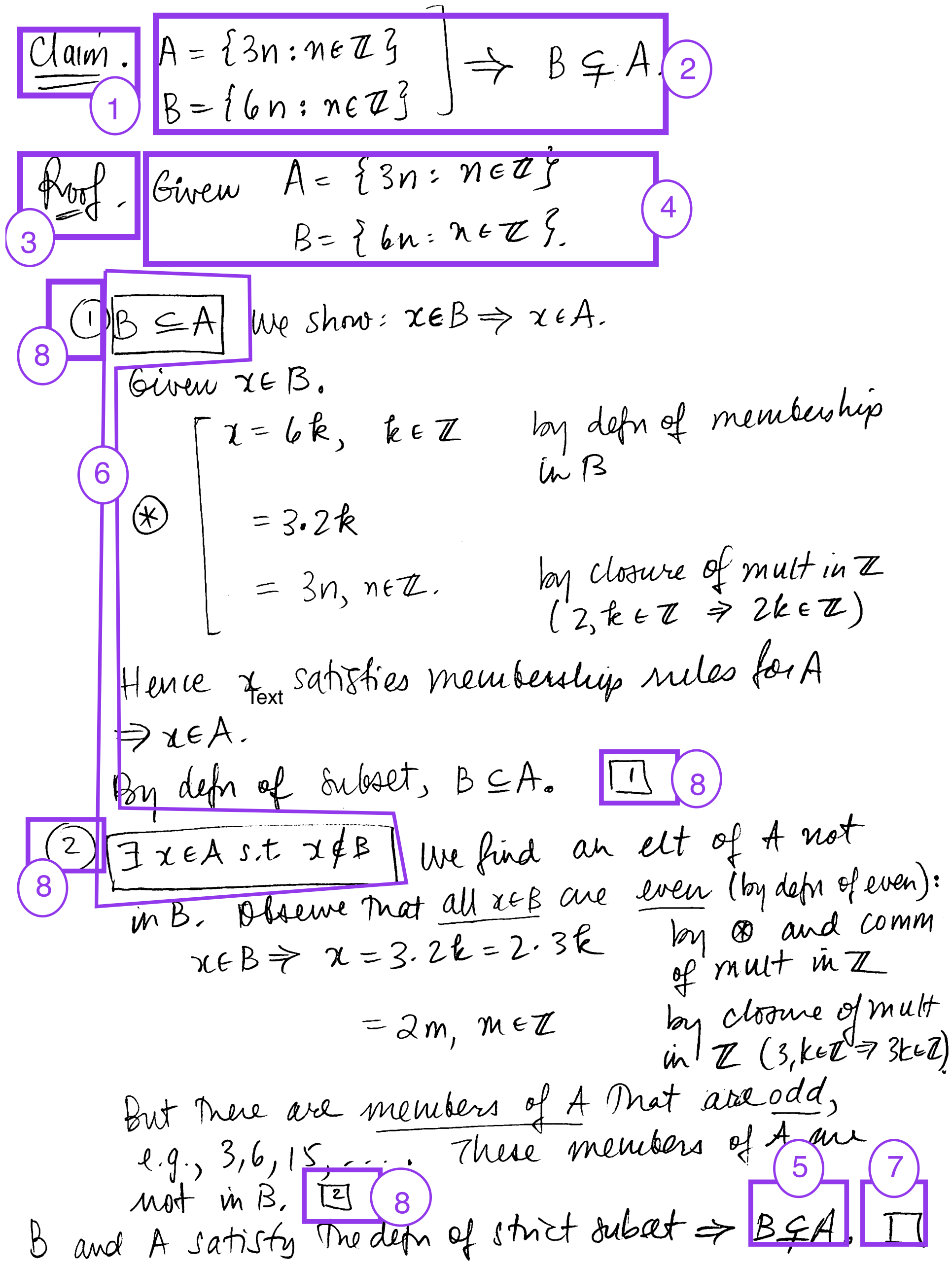
**Features of communicating proof well**

(**Essential features in bold**)

1. **Label the claim.**
2. **State the claim precisely.**
3. **Label the proof beginning.**
4. Begin a proof by reminding yourself and readers of the starting point: the conditions of the claim.
5. **End the proof with where you need to go: the conclusions of the claim.**
6. Summarize your approach to the reader.
7. **Label the proof end.** A traditional way is to use a box.
8. **Write up parts within a proof properly**. **Label when they begin and end.**
   - Give them a name (e.g., Claim A) if it is a proof within a proof
   - **Use labels like** [$\Rightarrow$] **and** [$\Leftarrow$] **if doing an if and only if proof.**
9. Diagrams are good only if you explain what you are showing. Give a caption.

Q5. Where are each of these features in the student's proof? Are there any features not present?

**Activity 5 — A Sample Proof, Part 6 (Reference: Features of Proof Communication)**



**Instructor Note:.** For Questions 1 - 3, have groups work together to fill in the gaps, then have them write up their completed proofs on large sheets of paper for display. Have each group critique a different group's proof, annotating the presented proof to highlight key features, missing features, and any questions they might have for the group that wrote the proof. For each proof, engage in a class discussion about the annotations. (Don't rush this process; it may take most of a class period.)

**Activity 6 — Proving a Conjecture**

Now that you have had a chance to critique a proof, let's try creating one together. Consider the following conjecture. An outline of the proof for this conjecture is provided. Fill in the missing details.

- If $A = \{f : \mathbb{R} \to \mathbb{R}, | \, f(x) = a(x-h)^2 + v$ where $a, h, v \in \mathbb{R}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, | \, f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}\}$, then $A = B$.

*Proof.* Let $A$ and $B$ be the sets defined in the conjecture.

Q1. *Why $A \subseteq B$:* Let $f \in A$. This tells us [Fill this in]... Hence, we may conclude that $f \in B$. 1
[*Solution. This tells us $f(x) = a(x-h)^2 + v$. Now, if we expand the binomial portion of the formula, we see $f(x) = ax^2 - 2ahx + (v + ah^2)$; and it follows that $f$ satisfies the membership rule for the set B. Hence, we may conclude that $f \in B$.*]

Q2. *Why $B \subseteq A$:* Let $g \in B$. This tells us [Fill this in]... Hence, we may conclude $g \in A$. 2
[*Solution. This tells us $g(x) = ax^2 + bx + c$. Now, if we complete the square, we see $g(x) = a(x-h)^2 + v$, where $h = -b/(2a)$ and $v = c - b^2/(4a^2)$; and it follows that $g$ satisfies the membership rule for the set A. Hence, we may conclude that $g \in A$.*]

Q3. Explain why we can now conclude that $A = B$. 3
[*Solution. In Part 1, we showed that if $f \in A$, then $f \in B$. Consequently, we may conclude that $A \subseteq B$. In Part 2, we showed that if $g \in B$, then $g \in A$. Consequently, we may conclude $B \subseteq A$. Since $A \subseteq B$ and $B \subseteq A$, we must conclude that $A = B$, as desired.*]

□

**Instructor note.** Question 4 is optional. If this problem is used, we recommend assigning it as homework and giving students some class time working in groups to sketch a rough outline of the proof strategy before finishing the problem individually.

**Activity 6 Follow-up**

**Definition 1.4.** Let $A$ and $B$ be sets.

- The *intersection* of $A$ and $B$ is defined to be the set consisting of those elements that are members of $A$ AND are members of $B$. This special set is denoted by $A \cap B$.
- The *union* of $A$ and $B$ is defined to be the set consisting of those elements that are members of $A$ OR are members of $B$. This special set is denoted by $A \cup B$.

Consider the following conjecture.

- If $A$, $B$, and $C$ are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Q4 Construct your own proof of this conjecture.

[*Solution. This proof requires two steps. First, we will prove that* $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. *Second, we will prove that* $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

*To begin, suppose* $x \in A \cap (B \cup C)$. *The tells us* $x \in A$ *AND* $x \in B \cup C$. *Now, since* $x \in B \cup C$, *we know that either* $x \in B$ *OR* $x \in C$. *Let's consider both cases. First, suppose* $x \in B$. *This tells us that* $x \in A$ *AND* $x \in B$*; therefore we may conclude* $x \in A \cap B$. *Second, suppose* $x \in C$. *This tells us that* $x \in A$ *AND* $x \in C$*; therefore we may conclude* $x \in A \cap C$. *Consequently, we must conclude that* $x \in A \cap B$ *OR* $x \in A \cap C$*; therefore, we know* $x \in (A \cap B) \cup (A \cap C)$. *We have shown that every member of* $A \cap (B \cup C)$ *is a member of* $(A \cap B) \cup (A \cap C)$*; therefore, we may conclude that* $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

*Now, suppose* $y \in (A \cap B) \cup (A \cap C)$. *This tells us* $y \in A \cap B$ *OR* $y \in A \cap C$. *Let's consider both cases. First, suppose* $y \in A \cap B$. *This tells us that* $y \in A$ *AND* $y \in B$. *Second, suppose* $y \in A \cap C$. *This tells us that* $y \in A$ *AND* $y \in C$. *Since* $y \in B$ *OR* $y \in C$, *we know that* $y \in B \cup C$. *Therefore, since we also know* $y \in A$, *we must conclude that* $y \in A \cap (B \cup C)$. *We have shown that every member of* $(A \cap B) \cup (A \cap C)$ *is also a member of* $A \cap (B \cup C)$*; therefore, we may conclude that* $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .]

# Back to the opening inquiry

**Instructor note.** We recommend two options for this section, depending on what fits your teaching style best.

- **Option 1: Interactive lecture.** Using the structure suggested below by the parts of Activity 7, engage the class in an interactive lecture and make sure teachers are taking notes in the spaces provided, or separately in notebooks. The interactive lecture should allow for some degree of student participation; for example, you could provide the first line or two of the "solution" to a question, then have teachers fill in the rest. Afterwards, you might distribute the worksheets and assign filling them in for homework.
- **Option 2: Distribute the parts of Revisiting the Parent Relation.** For the most part, the questions serve to "flesh out" a nontrivial proof. We recommend that teachers work together in small groups on these questions, pausing after each one to engage in a brief class discussion.

**Activity 7 — Revisiting the Parent Relation, Part 1**

Having devoted several activities to set concepts and proofs, let's take a second look at the Parent Relation from Activity 1.

**Number Parent Relation.** Two positive integers $a$ and $b$ are *parents* of another positive integer $c$ if $c = ab$. We will call $c$ the *child* of $a$ and $b$. A child cannot be one of its own parents.

From now on, let's say that a single number $n$ can count as a pair of parents for a child $c$ provided we have $n^2 = c$.

Let's also agree to say that an integer factor $p$ of an integer $a$ is *proper* provided $p \neq \pm 1$ and $p \neq \pm a$. In Activity 1, we made a conjecture, namely

> If a positive integer has least three different proper factors, then it has more than one pair of parents.
> There are no other numbers with more than one pair of parents.

**Applying set notation.** Using set notation, we can interpret the conjecture in the following way.

**Conjecture 1.5** (Number parent conjecture, take 1)**.** If we let

$$\begin{aligned} S &= \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\} \\ T &= \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\} \end{aligned}$$

then $S = T$.

Q1. How does this way of phrasing the conjecture match up with the original way?

(a) Look up the definition of set equality. What does $S = T$ mean by definition of set equality? [*Solution. By definition, $S = T$ means $S \subseteq T$ and $T \subseteq S$.*]

(b) Which part of set equality implies the first sentence ("If a number has least three different proper factors, then it has more than one pair of parents.")? [*Solution. Based on the defining rules for the sets, the expression $S \subseteq T$ means "If a number has least three different nontrivial factors, then it has more than one pair of parents."*]

(c) Which part of set equality implies the second sentence? ("There are no other numbers with more than one pair of parents") [*Solution. The expression $T \subseteq S$ means "there are no other numbers with more than one pair of parents."*]

**Activity 7 — Revisiting the Parent Relation, Part 2**

There is another mathematically equivalent way of phrasing the conjecture.

**Conjecture 1.6** (Number parent conjecture, take 2)**.** Let $n$ be a positive integer. The number $n$ has more than one pair of parents *if and only if* $n$ has at least three proper factors.

In mathematics, the phrase "if and only if" indicates the presence of a *biconditional* statement — a kind of "two for one" conditional statement. In particular, if $P$ and $Q$ represent logical statements, then "$P$ if and only if $Q$ represents the compound statement

- (If $P$ then $Q$) AND (If $Q$ then $P$).

Mathematicians often use "iff" (or the symbol $\iff$) as an abbreviation for the phrase "if and only if".

Proving an "if and only if" conjecture requires us to verify that *both* conditional statements are unbreakable promises.

Q2 Suppose $P$ and $Q$ are logical statements, and think about the conjecture $P \iff Q$. Explain why proving this conjecture establishes that $P$ and $Q$ are *logically equivalent*; that is, have the same truth value.

[*Solution. Proving $P \iff Q$ requires us to show that both conditional statements "If P then Q" and "If Q then P" cannot be FALSE. Now, a conditional statement can only be FALSE when its hypothesis is TRUE but its conclusion is FALSE. Once we know that "If P then Q cannot be FALSE, we also know that when P is TRUE, we must have Q TRUE as well. Once we know that "If Q then P" cannot be FALSE, we also know that when Q is TRUE, we must have P TRUE as well. Thus, P and Q must have exactly the same truth value — either both are TRUE or both are FALSE.*]

**Activity 7 — Revisiting the Parent Relation, Part 3**

Here are the two equivalent ways of phrasing our number parent conjecture, written side by side.

**Proposition 1.7** (Number parent proposition)**.**

| | |
|---|---|
| *If $S = \{n \in \mathbb{N} : n$ has at least three different non-trivial factors$\}$ and $T = \{n \in \mathbb{N} : n$ has more than one pair of parents$\}$, then $S = T$.* | *For all $n \in \mathbb{N}$, $n$ has more than one pair of parents if and only if $n$ has at least three different proper factors.* |

*Proof.* For readability, let $P$ represent the statement "The positive integer $n$ has more than one pair of parents" and let $Q$ represent the statement "The positive integer $n$ has at least three different proper factors."

**Part A: Proving the conjecture "If $P$ then $Q$".**

Suppose that $P$ is TRUE. In other words, suppose $n$ is a positive integer that has more than one pair of parents. Let $\{a, a'\}$ and $\{b, b'\}$ be two different pairs of number parents for $n$. We need to explain why we can conclude that $n$ has at least three different proper factors.

Q3 By assumption, we know $a \cdot a' = n = b \cdot b'$.
Explain why it is not possible to have $a = b$.
[*Solution. If $a = b$, then we have $a \cdot a' = a \cdot b'$. This forces us to conclude that $a' = b'$. However, this tells us $\{a, a'\} = \{b, b'\}$ — contrary to assumption.*]

Assuming that $\{a, a'\}$ and $\{b, b'\}$ are *different* pairs of number parents leads to several possibilities. We will have to consider each as a separate case.

**Case 1: Suppose $a \neq a'$ and $b \neq b'$.**

(a) Explain why it is not possible to have $a' = b$.
[*Solution. We know that $n = a \cdot a'$ and $n = b \cdot b'$. Now, if $a' = b$, when we would know $a \cdot b = b \cdot b'$. This forces us to conclude that $a = b'$. However, this again tells us that $\{a, a'\} = \{b, b'\}$ — contrary to assumption.*]

(b) What can we say about the number of proper factors for $n$?
[*Solution. We now know that $a \neq b$ and $b \neq a'$; hence, $a, a'$, and $b$ are all different proper factors of $n$. Therefore, $n$ must have at least three different proper factors.*]

**Case 2: Suppose that $a = a'$.**

(a) Explain why it is not possible to have $b = b'$. [*Solution. If $b = b'$, then we must conclude that $n = b^2$. However, since $a$ and $b$ are positive integers by assumption, this forces us to conclude that $b = \sqrt{n} = a$. We have already established that this is not possible.*]

(b) What can we say about the number of proper factors for $n$? [*Solution. We have established that $a \neq b$ and $b \neq b'$; hence, we must conclude that $n$ has at least three different proper factors.*]

(c) Explain why we do not need to consider the case where $b = b'$ or the case where $a' = b'$. [*Solution. The case where $b = b'$ is really the same as the case where $a = a'$. The case where $a' = b'$ is impossible.*]

We have now established that *IF* a positive integer $n$ has at least two different pairs of number parents, *THEN* $n$ has at least three different proper factors. A

**Activity 7 — Revisiting the Parent Relation, Part 4**

Part B: Proving the conjecture "If $Q$ then $P$."

Q4 Now, suppose that $Q$ is TRUE; that is, suppose $n$ is a positive integer that has at least three different proper factors. Let $a$, $b$, and $c$ be such factors. Either $a \cdot b = n$ OR $a \cdot b \neq n$. Let's consider each case.

(a) If$a \cdot b \neq n$ explain why we must conclude that $n$ has at least two pairs of number parents. [*Solution. We know that a and b are different proper factors of n. Hence, there exist different positive integers x and y such that* $a \cdot x = n$ *and* $b \cdot y = n$. *By definition,* $\{a, x\}$ *and* $\{b, y\}$ *are different pairs of parents for n.*]

(b) If $a \cdot b = n$, explain why it is not possible to have $a \cdot c = n$ as well. [*Solution. If* $a \cdot b = a \cdot c$, *then we must conclude* $b = c$ *— contrary to assumption.*]

(c) If $a \cdot b = n$, explain why we know that $n$ has at least two pairs of number parents. [*Solution. Since* $a \cdot c \neq n$, *we know there exists a positive integer* $x \neq a$ *such that* $c \cdot x = n$. *By definition,* $\{a, b\}$ *and* $\{c, x\}$ *are different pairs of number parents for n.*]

We have now established that *IF* a positive integer $n$ has at least three different proper factors, *THEN* $n$ has at least two different pairs of number parents. B

We have shown both directions, so we have shown that for all $n \in \mathbb{N}$, $n$ has more than one pair of parents if and only if $n$ has at least three different proper factors. □

# Summary of mathematical practices

**Instructor note.** Distribute this summary sheet after completing Activity 7. Go over the ideas presented and invite discussion. This summary is really just for reference purposes.

**Summary of Mathematical Practices**

## Clarifying the question

- Make the best sense as you can of the question with what is available.
- Identify what is unambiguous, and then identify what is ambiguous.
- For the ambiguous parts, play around with different possibilities to see what is the most mathematically interesting possibility. Sometimes you may find that there are multiple interesting mathematical possibilities.

## Conjecturing and Claim making

- Think of conejctures as "promises".

  If you're making a promise, you would want to make absolutely sure that everyone knows exactly what the promise means, and also that everyone would agree on what evidence would count as showing you have kept your promise!

  The same is true about mathematical statements. A mathematical statement needs to be crystal clear about what it means.
- Mathematical conjectures should either be true or false; if they "depend" on something, this means that there is often a better conjecture that can be made.
- The more general a conjecture, the greater its potential usefulness.

  For instance, "12 has more than one pair of parents" is a true conjecture, but a potentially more useful conjecture is "All numbers with at least three distinct proper factors have more than one pair of parents".
- The more "directions" a conjecture addresses, the more useful it will likely be.

  For instance, "All numbers with at least three distinct factors have more than one pair of parents" is a true conjecture, but "A number has more than one pair of parents if and only if it has at least three distinct factors" provides much more understanding.

## Exploring math: Our expectations

- Make a conjecture.
- Try to prove it.
- If you get stuck, consider the negation of the conjecture.
- Try to prove the negation instead.
- Consider the "opposite direction" conjecture. (The "converse" of the conjecture.)
- Try to prove it instead.
- Aim to make the most satisfying conjecture possible.
- Rewrite, rewrite, rewrite! Use the rewriting process to help things get clear for yourself, your future students, and your future self and peers.