Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

# MODULE($S^2$): Geometry for Secondary Mathematics Teaching

## Module 2: Transformational Geometry

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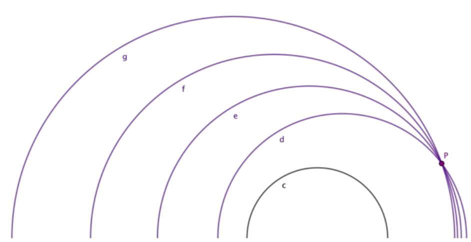



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# Contents

# MODULE($S^2$) Geometry Course Overview

In order to teach geometry one needs to understand its place in the wider mathematical context, its development and possible approaches to its study. There is much that can be done with and learned about geometry, but our path for this course will be focused on the needs of our students as future teachers.

We are used to the Euclidean approach to developing geometry involving an axiomatic approach and sometimes that of straightedge and compass constructions. Although we understand the need for our students to understand the axiomatic structure of geometry and to develop their proving skills, we also see competing needs for the time in a geometry class for preservice teachers. For many of our teachers, the geometry course they will be teaching necessitates understanding of transformations that are not always included in a traditional Euclidean geometry class.

For this reason we chose to split the course into three modules hoping to address content relevant to the practice of teaching mathematics.

- Module 1: Axiomatic Development
- Module 2: Transformational Geometry
- Module 3: Similarity

The work of a teacher consists of more than just possessing the knowledge of mathematics. Teachers are supposed to help others attain knowledge they are required to attain. For example, the teacher is tasked with creating, selecting, and modifying tasks; a skill not often addressed in the mathematics classroom. Another relevant skill is ability to understand, find flaws, and help students find errors in their arguments. In this course we are providing opportunities for students to develop and practice these skills in addition to learning geometry.

## Instructional Notes and Expectations

In effort to create a learning environment that is aligned with current recommendations for teaching and learning mathematics in K-12 settings, the following practices are suggested for instruction in this course:

- Use of Technology: We encourage the investigative use of dynamic geometry programs for the exploration of ideas generated in class and through assignments. In addition, there are places where students will be directed to particular internet sites to read about the concepts of study. However, we discourage students from general internet searches when completing homework and instead encourage them to problem solve and communicate with peers in order to work as mathematicians work when pursuing new ideas. Reminder of this policy is often necessary throughout the course.
- Note-taking Assignment: As there is no textbook for this course, we recommend assigning note-taking as part of the regular work of the course. Each day, one student should be assigned as the official note-taker, who then submits notes completed in a template you provide. These notes can be used as part of a homework or participation grade for the course and posted in your learning management system for all students to have access. If you use a template for this activity, it is easily combined into a book authored by your class at the end of the semester.
- Handouts: In-class activities for students and homework assignments are listed as handouts. However, most can be easily incorporated into a digital display for the class or shared electronically through your learning management system.
- Homework: In many cases, homework assignments are structured so that they generate discussion for the next or an upcoming lesson. Assigning homework so that it can be submitted through a learning management system prior to the class in which it will be discussed provides the instructor an opportunity to peruse the work and adequately anticipate questions for the following class discussions. In some cases, considering how students respond to questions in homework prior to a class session can aid in assigning students to groups that will then move forward in their thinking based on shared ideas.
- Video Assignments: Two of the culminating assignments require access to online videos. In order to engage with video animations of mathematics classrooms you will need to invite students to open accounts at specified websites.

## Equitable Teaching Practices in MODULE($S^2$) Curriculum

The curriculum produced by the Mathematics of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE($S^2$)) Project is an outgrowth of the work of the Mathematics Teacher Education Partnership (MTE-Partnership), a national collaborative working towards improving the number and quality of secondary mathematics teachers prepared in institutions of higher education. The MTE-Partnership is founded on a set of Guiding Principles (MTE-Partnership, 2014) which include a focus on transforming preparation programs so that prospective teachers develop teaching practices that demonstrate a dedication to equitable pedagogy (p. 5) and convey views that mathematics is a living and evolving human endeavor (p. 4). Equitable practice must include both appropriate equitable teaching practices and the development of learners mathematical identities.

As such, the MODULE($S^2$) curriculum materials include opportunities to engage in the use of teaching practices that support developing content knowledge in equitable ways as well as practices that support developing productive dispositions and identities in mathematics. The Mathematics Teaching Practices (NCTM, 2014) describe quality instructional practices with a focus on developing content knowledge of learners. This core set of eight research-based teaching practices support equitable teaching and are shown in the framework below (Figure 1). When implemented intentionally as interconnected practices in mathematics teaching, these practices provide space for the instructor to view all students as mathematical thinkers and to develop agency among the learners in the classroom (Berry, 2019).

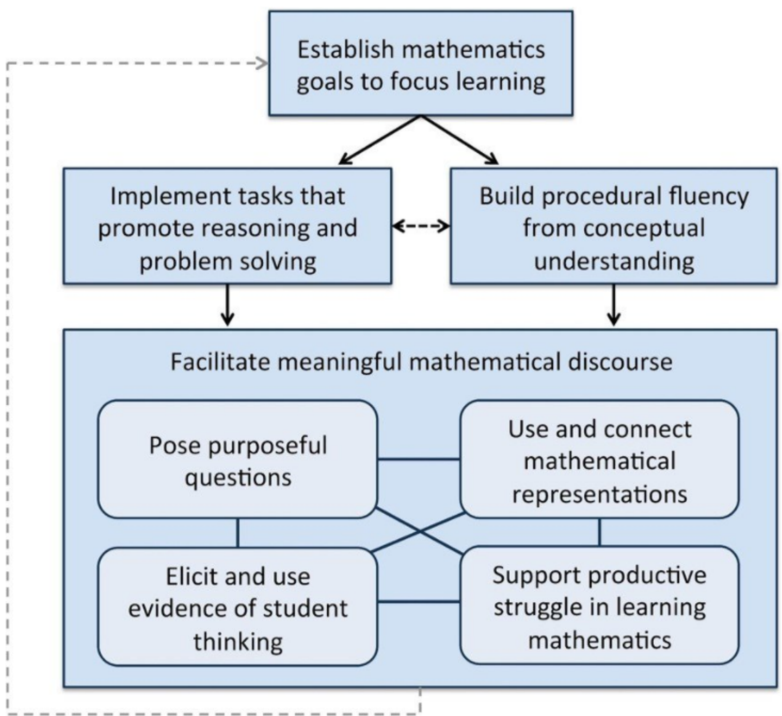


**Figure 1:** The Mathematics Teaching Framework demonstrating the interconnected nature of teaching practices that support equitable teaching. (Boston, Dillon, Smith, & Miller, 2017, p. 215).

In addition to implementing practices which focus on building content knowledge, the MODULE($S^2$) materials are intended to strengthen mathematical learning and cultivate positive student mathematical identity (Aguirre, Mayfield-Ingram, & Martin, 2013, p. 43). Throughout the MODULE($S^2$) materials, instructors will find opportunities to enact the following five equity-based practices:

| *Equity Based Practices That Attend to Learners Identities* |
| --- |
| **Go deep with mathematics.**<br>• Support students in analyzing, comparing, justifying, and proving their solutions.<br>• Engage students in frequent debates.<br>• Present tasks that have high cognitive demand and include multiple solution strategies and representations. |
| **Leverage multiple mathematical competencies.**<br>• Structure student collaboration to use varying math knowledge and skills to solve complex problems.<br>• Present tasks that offer multiple entry points, allowing students with varying skills, knowledge, and levels of confidence to engage with the problem and make valuable contributions. |
| **Affirm mathematics learners identities.**<br>• Promote student persistence and reasoning during problem solving.<br>• Encourage students to see themselves as confident problem solvers who can make valuable mathematical contributions.<br>• Assume that mistakes and incorrect answers are sources of learning.<br>• Explicitly validate students knowledge and experiences as math learners.<br>• Recognize mathematical identities as multifaceted, with contributions of various kinds illustrating competence. |
| **Challenge spaces of marginality.**<br>• Center student authentic experiences and knowledge as legitimate intellectual spaces for investigation of mathematical ideas.<br>• Position students as sources of expertise for solving complex mathematical problems and generating math-based questions to probe a specific issue or situation.<br>• Distribute mathematics authority and present it as interconnected among students, teacher, and text.<br>• Encourage student-to-student interaction and broad-based participation. |
| **Draw on multiple resources of knowledge (math, culture, language, family, community).**<br>• Make intentional connections to multiple knowledge resources to support mathematics learning.<br>• Use previous mathematics knowledge as a bridge to promote new mathematics understanding.<br>• Tap mathematics knowledge and experiences related to students culture, community, family, and history as resources.<br>• Recognize and strengthen multiple language forms, including connections between math language and everyday language.<br>• Affirm and support multilingualism. |

**Table 1:** Adapted from Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K8 mathematics: Rethinking equity-based practices.* Reston, VA: National Council of Teachers of Mathematics.

As instructors engage with the MODULE($S^2$) curriculum materials, efforts should be made to utilize equitable teaching practices in order to engage prospective teachers in learning about mathematics and learning about teaching in equitable ways. Through the activities and practices contained in the MODULE($S^2$) materials, instructors and students will have opportunities to reflect on the power dynamics inherent in the teaching and learning of mathematics and consider how that reflection might inform their practice. Instructors can find more detailed descriptions of these practices in the first three references below.

Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). The impact of identity in K8 mathematics: Rethinking equity-based practices. Reston, VA: National Council of Teachers of Mathematics.

Berry, R. Q. (2019, May). Presidents message: Examining equitable teaching using the mathematics teaching framework. National Council of Teachers of Mathematics. Retrieved from https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Robert-Q_-Berry-III/Examining-Equitable-Teaching-Using-the-Mathematics-Teaching-Framework/

Boston, M. D., Dillon, F., Smith, M. S., & Miller, S. (2017). Taking action: Implementing effective mathematics teaching practices in grades 912. Reston, VA: National Council of Teachers of Mathematics.

Mathematics Teacher Education Partnership. (2014). Guiding principles for secondary mathematics teacher preparation. Washington, DC: Association of Public Land Grant Universities. Retrieved from http://www.aplu.org/projects-and-initiatives/stem-education/SMTI_Library/mte-partnership-guiding-principles-for-secondary-mathematics-teacher-preparation-programs/File

National Council of Teachers of Mathematics (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: Author.

## Module II

# Transformational Geometry

In Module 2 our goal is to develop a solid understanding of distance-preserving transformations. The sequence of activities is organized so that the students are led to construct definitions, then use those definitions to develop properties of transformations, all the while taking their distance-preserving properties as axioms. In the lessons students will encounter common conceptions learners have about transformations and discuss ways of helping learners understand the nature of isometries. The transformational approach offers an opportunity to engage in cross-curricular development; computer science and coding fits well with geometry work. This gives another compelling reason to take this approach in secondary classrooms.
In this module we will have more opportunities to engage students with proof. We will take the time to prove that reflections generate the group of Euclidean isometries. We will classify isometries based on their fixed points and use this classification to find the generating set for the group of isometries. We complete the modules with a conversation about triangle congruence, which has for decades been the cornerstone of high school geometry.

- **Big ideas** distance-preserving transformations of the Euclidean plane come in three flavors: reflections, translations, and rotations, although we could easily work only with reflection as every isometry can be expressed as a composite of reflections.
- **Goals for studying the topic**:
  - Know the definitions of isometries (rigid transformations) and be able to use coordinates to describe them.
  - Be able to describe axioms in which we postulate that reflections, rotations, and translations are distance-preserving and prove theorems using this set of axioms.
  - Be able to describe the compositions and decomopsitions of isometries.
  - Know that reflections generate the group of isometries of the plane. Further, be able to explain that any isometry can be expressed as a composition of at most three reflections.
  - Know the definition of congruent triangles using isometries.
  - Be able to describe symmetries of plane figures.
  - Be able to prove SAS, SSS, ASA.
- **Rationale** The CCSS have changed the emphasis from the Euclidean approach, often using the SMSG set of axioms, to the transformational approach. At this point, the students have very little experience with transformations, especially in the more formal, axiomatic way. In this module, students will develop deeper understanding of isometric transformations, their properties, and their relationship to each other. They will see how the congruence theorems follow from the set of axioms built on transformations.
- **Connections to Secondary Mathematics** Transformations have become the building block of geometric reasoning in the secondary schools. The transformations are used to define congruence between geometric figures and students should be able to describe how the congruence theorems and corollary (SSS, SAS, ASA, AAS) follow from this definition.
- **Overview of content**
  - Lesson Introduction to Transformations: In this lesson, the students are asked to find a shortest path between two points given certain constraints. There are many different approaches to the solutions available, but one that is particularly fruitful involves transformations.
  - Lesson Distance-preserving Transformations: In this lesson the students will develop further their skill to write definitions. They will develop definitions for reflections, rotation, and translations, and discuss their usability. We will take the time to introduce and discuss the choices made by the writers of CCSS for mathematics in the way they are suggesting geometry is developed, and consider some historical developments of geometry curriculum.
  - Lesson Rotations and Reflections: Students take the time to perform rotations and reflections using definitions they developed. They analyze learners' thinking and work on developing their ability to assist learners in advancing their understanding of mathematical concepts.

- Lesson Transformations and Congruence: In this lesson we define congruence. For two congruent shapes, we find an isometry (a sequence or rigid motions) taking one to the other.
- Lesson Fixed Points: In this lesson, we further develop the understanding of congruence from a transformational perspective by first conjecturing and then proving properties about fixed points in isometries. This lays the foundation for understanding that the isometries are a group generated by reflections.
- Lesson Triangle Congruence: The culminating lesson in this module provides an opportunity for students to develop intuitions about triangle congruence, and then prove the triangle congruence theorems.
- Lesson Optional Explorations: Additional optional explorations are provided that delve into coordinate geometry, graph transformations, and slopes as related to transformational geometry. We also provide options for a topic presentation project that may serve as a culminating project for the course.

- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the *Standards of Mathematical Practice* as outlined in the CCSS, in this module we will particularly pay attention to:

SMP 1 Make sense of problems and persevere in solving them

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 4 Model with mathematics

SMP 5 Use appropriate tools strategically

SMP 6 Attend to precision

SMP 7 Look for and make use of structure

- **Expectations for assignments**
  - The students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.
  - Writing Assignments are to be graded and feedback provided. These assignments may be commented on before a final version is submitted for evaluation.
  - Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.
  - Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: ”An important part of doing mathematics is to learn to ask one’s own questions. You might not be able to answer them, but you can always learn more through seeking the answer to your questions. Which questions has this work made you ask? Record all the questions that you’d like to investigate on the class website.”

| Lesson | Projected Length | Geometric Content | In-Class Activities | Homework | Connections and Notes |
|---|---|---|---|---|---|
| 1: Introduction to Transformations | 180 min | Mathematical modeling, Transformations as functions | Activity: What is the Shortest Way?<br><br>Visualizing Functions HW Discussion<br><br>Handout: Standards for Mathematical Practice | Homework: Visualizing Functions<br><br>Homework: Writing Definitions | • We provide an alternate context for the Activity: What is the Shortest Way? task that allows for discussions of equity and social justice.<br>• Homework: Visualizing Functions assignment must be completed so that students can engage in the second class session discussion.<br>• Homework: Writing Definitions is used to begin the discussion for Lesson 2. Work must be collected from students prior to the start of Lesson 2. |
| 2: Distance-preserving Transformations | 90 min | Definition of reflection, rotation, and translation, introduction to geometry from a transformational perspective | Activity: Isometries Preserve Distance Handout: UCSMP<br><br>Axioms Handout:<br><br>Core Congruence | Formal Writing Assignment: Composing Isometries | • Student responses from Lesson 1 homework will be used to generate class discussion<br>• Students will begin work on a project that is best completed with a partner (Formal Writing Assignment: Composing Isometries) and will seed discussion in Lesson 4 |
| 3: Rotations and Reflections | 180 min | Constructing reflections and rotations, understanding student thinking, Introduction to CCSS Transformational Geometry | Activity: Reflections<br><br>Activity: Student Thinking about Rotations<br><br>Activity: Rotations | Homework: Simulation of Practice Written Assignment<br><br>Homework: Simulation of Practice Video Assignment | • The in-class activities provide introduction to the two homework assignments.<br>• Students should complete the Formal Writing Assignment: Composing Isometries prior to the next lesson. Adjust deadlines for these homework assignments appropriately around the project. |
| 4: Transformations and Congruence | 180 min | Transformational definition of congruence, Sequence of transformations mapping congruent objects, Isometric transformations as a group | Activity: How to Get From Here to There | Homework: When Does Order Not Matter?<br><br>Optional Homework: Symmetries | • You can choose what to emphasize (constructions, proofs, etc.) and split the class sessions as appropriate.<br>• The optional homework (Optional Homework: Symmetries) is the only place symmetry is treated in the modules so you may choose to assign it instead of Homework: When Does Order Not Matter?.<br>• It may be appropriate to assign Project: Presenting Geometric Topics as a culminating activity for the course at this time. Examples are provided in the final section of the module. |
| 5: Fixed Points | 90 min | Properties of isometries (conjecturing and proving), isometries as a group generated by reflections | Activity: Fixed Points Theorems<br><br>Activity: Fixed Points Theorems Proof Outlines | | • Students may be working on Project: Presenting Geometric Topics at this time.<br>• Homework from Lesson 4 may be carried through Lesson 5 if needed.<br>• The opening investigation found in Lesson 6, Activity: Construct This Triangle, may be assigned as homework to support the introduction to the next lesson. |
| 6: Triangle Congruence | 90 min | Development and proof of triangle congruence theorems in transformational geometry | Activity: Construct This Triangle<br><br>Activity: Triangle Congruence Theorems<br><br>Optional: Triangle Congruence Proofs | Formal Writing Assignment: Transformations | • Formal Writing Assignment: Transformations provides scaffolding for the proof activity if needed for your students.<br>• You will need to prepare measurement cards for each student for Activity: Construct This Triangle. |
| 7: Optional Explorations | Adjustable Length | Coordinate Geometry, Graph Transformations, Slopes | | Exploration 1: Coordinate Geometry<br><br>Exploration 2: Graph Transformations<br><br>Exploration 3: Slopes<br><br>Project: Presenting Geometric Topics | • These explorations are offered as options for delving more deeply into topics closely related to the study of transformational geometry. You may choose to use these as whole class activities or individual homework assignments. We also provide Project: Presenting Geometric Topics description for use as a culminating course project should you choose to do so. |

# 2 Distance-preserving Transformations

## Overview

### Length

1 Class Meeting, ~ 90 minutes

### Summary

Building on the definitions of transformation and isometric transformation reached in Lesson 1, Lesson 2 is devoted to analyzing definitions and arriving at precise definitions of translation, reflection, and rotation.
We spend time discussing and building operational definitions based on students' thinking from the homework. By doing so, we emphasize the importance of precision in geometric language along with reviewing the role of definitions in an axiomatic system. The lesson continues from these definitions to examine the transformational view of geometry set forth in the Common Core State Standards.

### Goals

- Students will analyze and refine definitions of reflection, rotation, and translation.
- Students will draft proofs showing that reflection, rotation, and translation are isometries.
- Students will compare mathematics teaching standards on transformational geometry to generated proofs of isometries to note differences.

### Connection to Standards

| **CCSSM Standard** | **Connection to Lesson** |
|---|---|
| MP3 Construct viable arguments and critique the reasoning of others. | Students will generate proofs that reflection, rotation, and translation are isometries. The level of rigor of the proof is left up to the instructor, but discussion of reasoning and critiquing the ideas of others will take place as students generate a plan for the proof or a fully-formed proof. |
| MP6 Attend to precision. | Students will analyze definitions for reflection, rotation, and translation written by each other, with specific attention to precision of the language used and operational uses of the definitions. Mathematical language, including the defining of terms, is an important place in which to discuss precision in our practice. Students will have opportunity to do so in this lesson. |
| 8.G.A.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. | Although students will not completely reach the idea of congruence in this lesson, the work on proofs of isometries moves thinking toward the ideas of congruence and understanding congruence in a transformational approach to geometry. |
| HSG.CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. | Much of this lesson is spent on both generating precise definitions of transformations (as built from students' provided definitions) and understanding the definitions of transformations as based upon other geometric objects for which we already have definitions. In this way, we begin to build an approach to transformations that is within an axiomatic system. |
| **Concepts Beyond CCSSM** | **Connection to Lesson** |
| Definitions as part of an axiomatic system. | The focus in this lesson is on building definitions precisely based on pre-defined terms and objects. This practice emphasizes precision and attention to the ways in which axiomatic systems are structured. |

## Materials

- Activity: Isometries Preserve Distance
- Handout: Core Congruence
- Handout: UCSMP Axioms
- Formal Writing Assignment: Composing Isometries

# Lesson Description

We begin this lesson with a discussion of the definitions for reflection, rotation, and translation that were generated in the previous homework. Working on definitions is an essential part of the study of geometry, particularly as we make sense of ideas as part of an axiomatic system. Learning to be precise in our language and write definitions so that they are operational and based on prior knowledge is part of the work of geometry. After the discussion of definitions, we turn to consider the ways in which transformational geometry is presented in curriculum, and how that might be different than some students think.

Discussion of Homework: Writing Definitions

1. If you ask a student for examples of these transformations, they are likely to tell you: slide, flip, and turn. What do they mean by that? If you ask them for a definition of these transformations, what might they say?
2. Give precise definitions of those transformations.

**Pedagogical Note.** Gather students' responses to the Homework: Writing Definitions with enough time to compile their responses to question 2 in a format you can display for the class. If you do not find enough variety in their responses, consider adding a few others of your own (i.e., unknown students' work). The goal is that the list does not contain a precisely correct definition but that you can use the definitions to seed a discussion about what is needed in order for a definition to be precise and useful.

I. Arrange students in small groups and display definitions that students submitted in their homework for question 2. Give small groups a few minutes to discuss the definitions that are displayed and analyze their usefulness. Then reconvene the whole class for a discussion on each transformation. (We recommend working with one transformation at a time and typically follow the order of reflection, rotation, and then translation.) During the discussion, carefully analyze their contributions and compare against their concept image of the specific transformation. Questions to ask during small group discussions are: Can I perform the transformation using your definition? Can I check if a transformation is of the required type using your definition?

   (a) It is not necessary for your class to establish these definitions, but we will use the provided definitions going forward in these materials. The arguments presented here can most certainly be modified to the way your class chooses to describe the isometries.

   - A reflection in line $\ell$ is a transformation of the plane which assigns to each point $P$ a point $P'$ such that:
     - $P' = P$, if $P$ is on $\ell$
     - $\ell$ is the perpendicular bisector of $\overline{PP'}$, if $P$ is not on $\ell$.
   - A rotation with center $O$ and angle $\alpha$ is a a transformation of the plane which assigns to each point $P$ a point $P'$ such that $\overline{OP} \cong \overline{OP'}$ and $\angle POP' \cong \alpha$.
   - A translation from $A$ to $B$ is a transformation of the plane which assigns to each point $P \neq A$ a point $P'$ such that $\overline{AB} \parallel \overline{PP'}$ and $d(A, B) = d(P, P')$, the image of $A$ is $B$, and $ABP'P$ is a parallelogram.

> **Note.** $A * B * C$ implies that $A$, $B$, and $C$ are collinear such that $B$ is between $A$ and $C$. See Module 1.

(b) It will be beneficial to compare these definitions to the ones the students found in the high-school geometry texts and discuss how appropriate these might be for the younger students. For those of you who may be wondering how are we suddenly talking about distance in our definition of translation, we provide a definition that does not use the notion of distance.

A translation from $A$ to $B$ is a transformation of the plane which assigns $A$ to $B$, and which assigns every other point $P$ (i.e., every point $P \neq A$) to a point $P'$ such that $\angle RPP' \cong \angle PAB$, for some $R$ such that $A * P * R$, $P'$ on the same side of $\overleftrightarrow{AP}$ as $B$, and $\overline{PP'} \cong \overline{AB}$.

> **Pedagogical Note.** When introducing this definition for translation, we have found it useful to encourage students to draw a diagram to accompany the definition in order to make sense of it.

Of course, this definition of a translation is rather convoluted and most certainly inappropriate for a ninth grader (although it provides nice connection to Hilbert's Axioms from Module 1). This may be a good time to discuss how appropriate definition is for use in a particular setting. While the latter definition may be entirely appropriate for a college student, for a student in ninth grade it is not.

(c) You can discuss the need for the definitions to be at the appropriate reading/comprehension level, while also being mathematically accurate. Help the students see the need to be mathematically honest: beware of using definitions that the students may need to unlearn later on. Facilitate this conversation by prompting them to discuss what someone would need to know about the students and their experiences in order to decide which definitions should be used within a learning community.

(d) The definition of rotations can be restated in such a way that it aids the high school student in remembering how to perform the rotation:

A rotation with center $O$ and angle $\alpha$ is a a transformation of the plane which assigns to each point $P$ a point $P'$ such that $P$ and $P'$ lie on the same circle around $O$ and that $\angle POP' \cong \alpha$.

> **Note.** Depending on the time, you may choose to demonstrate, or ask the students to demonstrate, how they would perform these particular transformations using definitions. This is the topic of the next class, and we choose not to do it here. We want to offer the students an opportunity to develop their intuition a little more through the discussions and assignments, and then engage them into construction. This sequencing will help the students realize the value in understanding and learning definitions.

II. Before moving onto the next discussion, we should give students a moment to reflect and synthesize the discussion for themselves. Take a moment to hand out:
Activity: Isometries Preserve Distance

> Include here any discussion points and observations you found significant during our work on definitions for the three types of transformations: reflections, rotations, and translations.
>
> Prove that these three transformations are isometries:
> Reflection
> Rotation
> Translation

Students should record their impressions as well as the class agreed definitions into the handout.

III. At this point we are ready to convince ourselves that these transformations indeed preserve distance. In order to start that discussion, we'd like to bring the students' intuition into the conversation. As a whole group, ask the students to list all the properties of these three transformations they think are important. Once you have a list, narrow it down to the list that all three have in common. Some things you may hear are:

- They preserve distance
- They preserve angle
- They send lines to lines
- They preserve betweenness

IV. Next engage small groups in a jigsaw proof task, proving the three transformations are isometries. The proofs are provided in the handout: Activity: Isometries Preserve Distance.

(a) We are not suggesting that you have to demonstrate these proofs for the students. Their experiences from the first module should allow them successful engagement with this task. We suggest splitting the class into three groups, each in charge of working on one of the three transformations. Once completed, regroup them, so that each new group has 3 members, one from each old group. They present the proofs to each other and offer/receive feedback on their presentations.

(b) Remind the students of their strategies from last class, but also point out that we don't have a generalized coordinate representation of each isometry, only the three specific ones from the assignment. They may suggest finding the algebraic representation for these isometries, and this may be something you are interested in pursuing. We insisted, instead, they use only axioms and theorems from Hilbert's set of axioms.

(c) Through this work the students should notice that they need to use congruence theorems. They should realize at this point that triangles are essential for proving congruence of segments or angles. The fact that corresponding parts of congruent figures are congruent (CPCFC from now on) is essentially our only recourse in proving congruence of segments and angles.

V. Reiterate, if necessary, that having all of the congruence theorems was necessary to justify that these three transformations send segments to congruent segments. However, there is still a question of why that implies that those segments have equal lengths. At this point we'll take a moment to check how the Core Standards advise we approach congruence. A brief look at the Handout: Core Congruence brings a bit of a surprise. Allow students some time to read through, discuss, and ask questions about this new set of axioms.

(a) The core takes a different approach: it uses distance-preserving transformations to define congruence. In other words, this requires a different set of axioms, one in which we declare reflections, rotations, and translations to be distance-preserving (Handout: UCSMP Axioms, for instance).

(b) We find it useful to include some historical remarks at this point:

> **Note.** The following is adapted from Chapter 4 of Secondary Mathematics for Mathematicians and Educators: A View from Above by Michael Weiss (Routledge; to be published in late 2020).

Hilbert and Birkhoff/SMSG represent two distinct approaches to filling in the gaps of Euclids axiomatization of Geometry. A third approach, developed in the late 19th century, takes transformations as the fundamental objects.

The role of transformations in geometry teaching has waxed and waned over the past hundred years, but there is little doubt that the way mathematicians conceptualize geometry has been radically transformed by the transformational perspective. In 1872 Felix Klein became a full professor at the University of Erlangen, where he launched what came to be known as the Erlangen programme  nothing less ambitious than a complete reorganization of geometry along the lines of what was just beginning to be understood as the beginnings of group theory. Klein proposed that geometry be understood as the study of those properties of a space (such as, for example, a Euclidean plane) that are left invariant under the action of some group of transformations. Different choices of the group of transformations lead, naturally, to different sets of invariant properties, and thus to different kinds of geometries. Thus, projective geometry, Euclidean geometry, hyperbolic (and other non-Euclidean) geometries each is realized as a set of invariants under a different group of transformations.

This transformation-based perspective was a radical departure from the synthetic approach of Euclid and his modern updaters, in which the goal was to build towers of theorems on a foundation of axioms and postulates. Kleins Erlangen programme was unlike Hilberts Grundlagen in that it did not seek to fill the logical holes in the foundations of Euclids Elements, but rather to replace the entire structure with a completely new approach.

The impact of Klein's transformation-based geometry on schools varied across different countries. It had its greatest impact in the European educational system, where over the course of the 20th century transformations were incorporated into the secondary curriculum in France, Germany and the former Soviet Union. In the United States, however, the transformation-based approach to geometry largely failed to penetrate the secondary curriculum until 1972, when Usiskin and Coxford published their textbook Geometry: A transformation approach. More recently, the CCSS in Mathematics have advocated for a more thorough incorporation of transformations into the secondary curriculum, and a number of Standards-aligned textbooks have followed suit.

VI. Upon completion of this lesson, students are ready to work with a partner on Formal Writing Assignment: Composing Isometries. In this assignment, students will explore compositions of transformations using dynamic geometry software. The ultimate goal of the exploration is for students to generate understanding of the isometries as a group generated by reflections. Although most will not reach this clear end, you should leave discussion of compositions of isometries until after students have had opportunity to think about these relationships for themselves. Allow at least one week to complete this assignment.

## Preparation for the Next Lesson

- Students:
  - Formal writing assignment: Composing Isometries
- Instructor:
  - Formal writing assignment: Composing Isometries (should be due before you reach Lesson 4).
  - Determine pairs which will complete the formal writing assignment together. It is assigned as group work to allow sharing of the ideas which we find generates more robust exploration and findings. Each pair will submit a single paper.

## Activity: Isometries Preserve Distance

Include here any discussion points and observations you found significant during our work on definitions for the three types of transformations: reflections, rotations, and translations.

Class established definitions:

Reflection

Rotation

Translation

Prove that these three transformations are isometries:

Reflection

Rotation

Translation

## Handout: Core Congruence

COMMON CORE STATE STANDARDS for MATHEMATICS

**Congruence G-CO**

**Experiment with transformations in the plane**

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

**Understand congruence in terms of rigid motions**

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

**Prove geometric theorems**

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

**Make geometric constructions**

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

## HANDOUT: UCSMP AXIOMS

The University of Chicago School Mathematics Project was founded in 1983 with the aim of upgrading mathematics education in elementary and secondary schools throughout the United States. They developed a set of axioms that are in wide use today, but are also redundant in the sense that some axioms can be proved from others. The purpose of the redundancy was to make the learning of geometry more intuitive. These axioms used incorporated a transformational approach. Details of the project's history is given on its web page at http://ucsmp.uchicago.edu/ history.html.
The only undefined terms are **point**, **line**, and **plane**.

**Point-Line-Plane Axioms**

Axiom 1 Through any two points there is exactly one line.

Axiom 2 Every line is a set of points that can be put into a one-to-one correspondence with the real numbers, with any point corresponding to zero and any other point corresponding to the number 1.

Axiom 3 Given a line in a plane, there is at least one point in the plane that is not on the line. Given a plane in space, there is at least one point in space that is not on the plane.

Axiom 4 If two points lie in a plane, the line containing them lies in the plane.

Axiom 5 Through three non-collinear points, there is exactly one plane.

Axiom 6 If two different planes have a point in common, then their intersection is a line.

**Distance Axioms**

Axiom 7 On a line, there is a unique distance between two points.

Axiom 8 If two points on a line have coordinates $x$ and $y$ the distance between them is $|x - y|$.

Axiom 9 If point $B$ is on the line segment $AC$ then $AB + BC = AC$, where $AB$, $BC$, $AC$ denote the distances between the points.

**Triangle Inequality**

Axiom 10 The sum of the lengths of two sides of a triangle is greater than the length of the third side.

**Angle Measure**

Axiom 11 Every angle has a unique measure from $0^o$ to $180^o$.

Axiom 12 Given any ray $\overrightarrow{VA}$ and a real number $r$ between 0 and 180 there is a unique angle $\angle BVA$ in each half-plane of $\overleftrightarrow{VA}$ such that $\angle BVA = r$.

Axiom 13 If $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are the same ray, then $\angle BVA = 0$.

Axiom 14 If $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are opposite rates, then $\angle BVA = 180$.

Axiom 15 If $\overrightarrow{VC}$ (except for the point $V$ ) is in the interior of angle $\angle AVB$ then $\angle AVC + \angle CVB = \angle AVB$.

**Corresponding Angle Axiom**

Axiom 16 Suppose two coplanar lines are cut by a transversal. If two corresponding angles have the same measure, then the lines are parallel. If the lines are parallel, then the corresponding angles have the same measure.

**Reflection Axioms**

Axiom 17 There is a one to one correspondence between points and their images in a reflection.

Axiom 18 Collinearity is preserved by reflection.

Axiom 19 Betweenness is preserved by reflection.

Axiom 20 Distance is preserved by reflection.

Axiom 21 Angle measure is preserved by reflection.

Axiom 22 Orientation is reversed by reflection.

**Area Axioms**

Axiom 23 Given a unit region, every polygonal region has a unique area.

Axiom 24 The area of a rectangle with dimensions $l$ and $w$ is $lw$.

Axiom 25 Congruent figures have the same area.

Axiom 26 The areas of the union of two non-overlapping regions is the sum of the areas of the regions.

**Volume Axioms**

Axiom 27 Given a unit cube, every solid region has a unique volume.

Axiom 28 The volume of a box with dimensions $l$, $w$, and $h$ is $lwh$.

Axiom 29 Congruent solids have the same volume.

Axiom 30 The volume of the union of two non-overlapping solids is the sum of their volumes.

Axiom 31 Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the intersections have equal areas, then the two solids have the same volume.

## FORMAL WRITING ASSIGNMENT: COMPOSING ISOMETRIES

So far you have spent some time getting to know the three isometries of the plane: translations, rotations, and reflections.
Have you ever stopped to wonder if there is some other isometry hiding out there? Who knows? Perhaps if we can combine known isometries in just the right way, we may discover some previously unknown isometry...

In this assignment, you will use Geogebra to look at what happens when we compose isometries. After some exploration, we will make conjectures and try to prove them. Remember that a picture is worth a thousand words. You'll want to keep the diagrams you produce so you can include them in your paper as supporting and clarifying evidence[3].

1. **Composing translations**

   (a) **Composing two translations:** Using your Geogebra software, create a polygon (we will refer to it as $poly_1$). Create two arbitrary vectors ($v_1$ and $v_2$). Translate $poly_1$ by $v_1$ to create $poly_1'$, then translate $poly_1'$ by $v_2$ to create $poly_1''$. You may hide $poly_1'$ at this time and focus your attention on $poly_1$ and $poly_1''$. We have composed two translations[4], which we will call $t_{v_1}$ and $t_{v_2}$, to create the transformation $T = t_{v_2} \circ t_{v_1}$ where $T(poly_1) = poly_1''$.
   You may experiment by adjusting $v_1$ and $v_2$ as much as you like. When you feel ready, answer the following:

      i. Describe $T$ as a single transformation, rather than a composite. Is it one of the ones we already know? Be specific in its description - you need to clear describe all the information for someone to be able to reproduce your transformation exactly.

      ii. Does the relationship between the two vectors change the outcome in any way? Explain why or why not.

   (b) **Composing multiple translations:** Is it worth investigating the result of composing 3 or more translations? Why or why not?

2. **Composing reflections**

   (a) **Composing two reflections:** Similarly to part 1, create a polygon and two arbitrary lines ($\ell_1$ and $\ell_2$). Reflect $poly_1$ over $\ell_1$ to create $poly_1'$, then reflect $poly_1'$ over $\ell_2$ to create $poly_1''$. You may hide $poly_1'$ at this time and focus your attention on $poly_1$ and $poly_1''$. We have composed two reflections, which we will call $r_{\ell_1}$ and $r_{\ell_2}$, to create the transformation $T = r_{\ell_2} \circ r_{\ell_1}$ where $T(poly_1) = poly_1''$.
   You may experiment by adjusting $\ell_1$ and $\ell_2$ as much as you like. When you feel ready, answer the following:

      i. Describe $T$ as a single transformation, rather than a composite.

      ii. Does the relationship between the lines matter (parallel, intersecting, perpendicular, coincident)?

   (b) **Composing three reflections:** Next add a third line, $\ell_3$. Reflect $poly_1''$ over $\ell_3$ to create $poly_1'''$. We will define a new transformation $T = r_{\ell_3} \circ r_{\ell_2} \circ r_{\ell_1}$, where $T(poly_1) = poly_1'''$.
   Again, you may experiment by adjusting $\ell_1$, $\ell_2$, and $\ell_3$, this time focusing your attention on $poly_1$ and $poly_1'''$. There are a lot more possibilities for our lines, so here is a checklist:

      - Can you describe the transformation when all lines are parallel? What if some lines are coincident?
      - How about when you move a line so that it is no longer parallel? Does it depend on which line you move?
      - How about when you make one of the lines perpendicular to the other two? Does it depend on which line you chose?
      - How about when they all intersect at a common point?
      - How about when every pair of lines intersects (without a common intersection)?

   **HINT:** If you get stuck, it might help to think about what we learned in part 2a about composing two reflections.

[3] You should use Export feature in Geogebra.

[4] Keep in mind that $t_{v_2} \circ t_{v_1}$ means that we are first performing the transformation $t_{v_1}$, then applying the transformation $t_{v_2}$ to the result.

(c) **Composing multiple reflections:** What would happen if we try to combine more reflections? Will we ever get something new? Why?

3. **Composing rotations**

   (a) **Composing two rotations:** Once again create a polygon. This time arbitrarily rotate it about a point (you will need to create an arbitrary angle and an arbitrary point). Then arbitrarily rotate the image about another point (you will need another arbitrary angle and point). To start you may want to use the same point as the center of both rotations, but make sure you investigate the more general case. Again, we suggest you adjust the angles and points to try and see the bigger picture.

      i. Describe $T$ as a single transformation, rather than a composite.

      ii. How do the angles and centers of rotation affect the transformation?

   (b) **Composing multiple rotations:** Is it worth investigating the result of composing 3 or more rotations? Why or why not?

4. **Decomposing:** So far you have worked on understanding compositions. This process may have provided some insights into decompositions as well.

   (a) Given a rotation, can you find two reflections that compose to produce that rotation?

   (b) Given a translation, can you find two reflections that compose to produce that translation?

5. **Mix and match:** You may have wondered why is it that we have not mixed and matched translations, rotations, and reflections into a single transformation. All of our work thus far indicates that reflections are the building blocks of isometries, so what would be the point? However, a composite of a particular reflection and translation has its own special name: *glide reflection*. We obtain a glide reflection by composing a reflection and a translation for which the line of reflection is parallel to the vector of the translation. Analyze glide reflection in the view of your work with composites of reflections only. Finally, give a method for determining the line of reflection and the vector of translation for a given glide reflection.

**Final product to be submitted:** What have you learned? You should write this as an article someone can learn from. Your article should communicate clearly your results and why those results are valid. Include diagrams, and justifications for your claims.